Exercise 3

1. Find the moment generating function of $N(\mu, \sigma^2)$. Find the mean and the variance of $Y = e^X$ if $X \sim N(\mu, \sigma^2)$. (Y has the Lognormal distribution, very popular as a skew distribution for positive variables.)

2. State and prove the Markov Inequality. Use it to establish that, if Y is a random variable with MGF $M_Y(t)$, then for $\gamma > 0$ and $t > 0$,

$$P(Y \geq \gamma) \leq \frac{M_Y(t)}{e^{\gamma t}}.$$

Hence show that if Z has a standard normal distribution, then for $\gamma \geq 0$,

$$P(Z \geq \gamma) \leq e^{-\gamma^2/2}.$$

3. (a) Let X be a random variable with MGF $M_X(t)$. Find an expression for the cumulant generating function $K_X(t)$ as a function of the moments m_r. Hence find expressions for κ_2 and κ_3 in terms of these moments.

   (b) Let $Y = X - m_1$. Find expressions for κ_2, κ_3, κ_4 and κ_5 as functions of the moments of Y.

   (c) Now let $Z = Y/\sigma$. Find an expression for $K_Z(t)$ as a function of $K_Y(t)$. Hence find κ_2, κ_3 and κ_4 for Z as functions of the moments of Y.

   (d) Finally, suppose that X is distributed $N(\mu, \sigma^2)$. Write down $K_X(t)$ in this case. Use your results in (a) to find m_2 and m_3 for X.

4. Find the moment generating function of the Double Exponential or Laplace distribution with density function

$$f_X(x) = \frac{1}{2} e^{-|x|} \quad -\infty < x < \infty,$$

and hence its first four cumulants. (It has been suggested that the Laplace distribution is more typical of naturally occurring physical measurements than the normal distribution.)

5. If $H(x)$ and $G(y)$ are distribution functions, for which of the following definitions is $F(x, y)$ a joint distribution function? (Sometimes one can give a simple description of the joint distribution of X and Y which leads to the form given. The notations $\max(a, b)$ and $\min(a, b)$ mean the largest of a, b and the smallest of a, b respectively.)

   (a) $F(x, y) = H(x) + G(y)$

   (b) $F(x, y) = H(x)G(y)$ (use independence)

   (c) $F(x, y) = \max[H(x), G(y)]$

   (d) $F(x, y) = \min[H(x), H(y)]$ (consider $X = Y$).

